New Exact Solutions and Fractal Localized Structures for the (2+1)-Dimensional Boiti-Leon-Pempinelli System

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In this work, a novel phenomenon that localized coherent structures of a (2+1)-dimensional physical model possess fractal properties is discussed. To clarify this interesting phenomenon, we take the (2+1)-dimensional Boiti-Leon-Pempinelli (BLP) system as a concrete example. First, with the help of an extended mapping approach, a new type of variable separation solution with two arbitrary functions is derived. Based on the derived solitary wave excitation, we reveal some special regular fractal and stochastic fractal solitons in the (2+1)-dimensional BLP system. — PACS: 05.45.Yv, 03.65.Ge

Key words: Extended Mapping Approach; Boiti-Leon-Pempinelli System; Fractal Soliton.

1. Introduction

Fractals and solitons are two important parts of nonlinearity, which have been widely applied in many natural sciences [1 – 4], particularly in almost all branches of physics such as fluid dynamics, plasma physics, field theory, nonlinear optics and condensed matter physics [5 – 7]. Conventionally, these two aspects are treated independently since one often considers solitons as the basic excitations of an integrable model while fractals are elementary behaviors of nonintegrable systems. In other words, one does not analyze the possibility of existence of fractals in a soliton system. However, the above consideration may not be complete, especially in higher dimensions. In our recent study of higher-dimensional soliton systems [8 – 17], we have found that some characteristic lower-dimensional arbitrary functions exist in exact solutions of certain higher-dimensional integrable models. This means that any lower-dimensional fractal solution can be used to construct an exact solution to a higher-dimensional integrable model, which also implies that any exotic behaviors such as chaotic behavior and/or fractal property may propagate along this characteristics. Actually, scientists have found abundant fractal solitons in (2+1)-dimensions [15].

Now an important and interesting question is: Are there, similar or new fractal localized structures in other higher-dimensional soliton systems? In other words, are fractals in higher-dimensional physical models quite universal phenomena? Meanwhile, as far as we know, all the previously found fractal solitons in (2+1)-dimensions were obtained by Bäcklund transformation and a special variable separation approach. Now a subsequent intriguing issue is whether the fractal soliton solution to a (2+1)-dimensional integrable system can also be derived by other method such as symmetry reduction method [18 – 20], mapping approach and so on [21 – 23].

To answer these questions, we take the (2+1)-dimensional Boiti-Leon-Pempinelli (BLP) system as a concrete example

$$u_{yt} - (u^2 - u_x)_{xy} - 2v_{xxx} = 0, \quad v_t - v_{xx} - 2uv_x = 0. \quad (1)$$

The integrability of the above BLP system was established in [24]. In [25], it was shown that the BLP system was Hamiltonian, and pointed out that by a certain transformation the sin-Gordon equation or the sinh-Gordon equation can be derived from the BLP model. Soliton-like and multisoliton-like solutions for this equation have also been discussed by Lan and Zhang [26]. In the following parts of the paper, we will discuss its new exact solutions and special novel fractal localized structures.

2. Extended Mapping Approach and New Exact Solutions of the (2+1)-Dimensional BLP System

As is well-known, to search for solitary wave solutions to a nonlinear physical model, we can apply different approaches. One of the most efficient methods of finding soliton excitations of a physical model is the so-called mapping transformation method. With the help of the mapping transformation idea and based on the general reduction theory, we extend the mapping approach. The basic idea of the algorithm is as follows: Consider a given nonlinear partial differential equation (NPDE) with independent variables $x = (x_0 = t, x_1, x_2, \cdots, x_m)$ and a dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{2}$$

where P in general is a polynomial function of the indicated arguments, and the subscripts denote the partial derivatives. We assume that its solution is in an extended symmetry form, namely

$$u = \sum_{i=-N}^{N} \alpha_i(x)\phi^i(q(x)), \tag{3}$$

with

$$\phi' - \phi^2 = \sigma, \tag{4}$$

where $\alpha_i(x)$, $q(x)$ are arbitrary functions to be determined, $x = (x_0 = t, x_1, x_2, \cdots, x_m)$, σ is a constant and the prime denotes the first derivative of function ϕ with respect to q. To determine u explicitly, one can take the following procedure: First, similar to the usual mapping approach, determine N by balancing the highest-order nonlinear term with the highest-order partial derivative term in the given NPDE. Second, substitute (3) and (4) into the given NPDE and collect the coefficients of polynomials of ϕ; then set to zero each coefficient to construct a set of partial differential equations for $\alpha_i(x)$ $(i = -n, ..., -1, 0, 1, ..., n)$ and $q(x)$. Third, solve the system of partial differential equations to obtain $\alpha_i(x)$ and $q(x)$. Finally, as (4) possesses the general solutions

$$\phi = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}q), & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}q), & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}q), & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}q), & \sigma > 0, \\ -1/q, & \sigma = 0, \end{cases} \tag{5}$$

substituting $\alpha_i(x)$, $q(x)$ and (5) into (3) one can derive the exact solutions of the given NPDE.

Now we apply their extended mapping approach to (1). By the balancing procedure, ansatz (3) becomes

$$\begin{aligned} u &= f + g\phi(q) + h\phi^{-1}(q), \\ v &= F + G\phi(q) + H\phi^{-1}(q), \end{aligned} \tag{6}$$

where f, g, h, F, G, H, and q are arbitrary functions of $\{x, y, t\}$ to be determined. Substituting (6) and (4) into (1) and collecting coefficients of polynomials of ϕ, then setting each coefficient to zero, yields

$$2Gq_x^3 + g^2 q_x q_y - g q_y q_x^2 = 0, \tag{7}$$

$$\begin{aligned} &-g q_y q_{xx} + g^2 q_{xy} - g_y q_x^2 + 2fg q_x q_y + 2g g_y q_x \\ &\quad + 2g q_y g_x - g q_y q_t - 2g q_x q_{xy} - 2g_x q_x q_y \\ &\quad + 6G_x q_x^2 + 6G q_x q_{xx} = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &8\sigma g^2 q_x q_y + 2s g_x q_y + 2g f_y q_x + 2g f_x q_y + 16G\sigma q_x^3 \\ &\quad - g_y q_{xx} + 2fg q_{xy} - g_t q_y - 2g_{xy} q_x \\ &\quad + 6G_{xx} q_x + 2fg_y q_x + 6G_x q_{xx} - g q_{yt} \\ &\quad + 2g_y g_x - 2g_x q_{xy} + 2G q_{xxx} - 8\sigma g q_y q_x^2 \\ &\quad + 2g g_{xy} - g q_{xxy} - g_{xx} q_y - g_y q_t = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &4\sigma q_x q_y (fg - g_x) + 2\sigma g q_{xy}(g - 2q_x) \\ &\quad + \sigma q_{xx}(12G q_x - 2g q_y) \\ &\quad + \sigma q_x^2 (12G_x - 2g_y) - g_{xxy} + 4\sigma g g_y q_x \\ &\quad + 2f_y g_x + 4\sigma g g_x q_y - 2\sigma g q_y q_t - g_{yt} \\ &\quad + 2fg_{xy} + 2f_x g_y + 2g f_{xy} + 2G_{xxx} = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &q_x q_y (2h^2 + 2\sigma^2 g^2 - 2\sigma^2 g + 2\sigma h q_x) \\ &\quad + 4\sigma q_x^3 (\sigma G - H) \\ &\quad + q_{xy}(2\sigma fg - 2\sigma g_x + 2h_x - 2fh) \\ &\quad + q_{xxy}(h - \sigma g) + q_{xxx}(2\sigma G - 2H) \\ &\quad + q_{xx}(h_y - 6H_x + 6\sigma G_x - \sigma g_y) \\ &\quad + q_x(2\sigma f g_y - 2f_y h - 2f h_y + 2\sigma g f_y \\ &\quad - 2\sigma g_{xy} - 6H_{xx} + 2h_{xy} + 6\sigma G_{xx}) \\ &\quad + q_{yt}(h - \sigma g) + q_y(2\sigma f g_x - 2h f_x \\ &\quad - 2f h_x + 2\sigma g f_x + h_t - \sigma g_{xx} \\ &\quad + h_{xx} - \sigma g_t) - \sigma g_y q_t + 2h_y g_x \\ &\quad + 2f_x f_y + h_y q_t + 2g_y h_x + 2f f_{xy} \\ &\quad + 2g h_{xy} + 2h g_{xy} - f_{xxy} + 2F_{xxx} - f_{yt} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\sigma q_{xx}(12H q_x - 2h q_y) - 2h q_{xy}(2\sigma q_x + h) \\ &\quad + 4\sigma q_x q_y (fh - h_x) + 2\sigma q_x^2 (6H_x - h_y) \\ &\quad - h_{xxy} - h_{yt} + 2h f_{xy} + 2H_{xxx} \end{aligned}$$

$$-2hq_y(\sigma q_t+2h_x)+2fh_{xy}+2h_yf_x+2f_yh_x-4hh_yq_x=0, \tag{12}$$

$$\begin{aligned}&2\sigma q_{xy}(h_x-fh)+\sigma q_{xx}(h_y-6H_x)\\&+8\sigma hq_xq_y(h+\sigma q_x)-2\sigma Hq_{xxx}\\&-16\sigma^2Hq_x^2+\sigma hq_{xxy}\\&+2\sigma q_x(h_{xy}-hf_y-fh_y-3H_{xx})\\&+\sigma q_y(h_{xx}-2fh_x-2hf_x+h_t)+2hh_{xy}\\&+\sigma h_yq_t+\sigma hq_{yt}+2h_yh_x=0,\end{aligned} \tag{13}$$

$$\begin{aligned}&\sigma^2q_{xx}(6Hq_x-hq_y)-h\sigma qxy(2\sigma q_x+h)\\&+2\sigma^2q_xq_y(fh-h_x)+\sigma^2q_x^2(6H_x-h_y)\\&-2\sigma h(h_yq_x+h_xq_y)-\sigma^2hq_yq_t=0,\end{aligned} \tag{14}$$

$$\sigma^3hq_x^2q_y+\sigma^2h^2q_xq_y-2\sigma^3Hq_x^3=0, \tag{15}$$

$$Gq_x^2+gGq_x=0, \tag{16}$$

$$2gG_x-Gq_t+Gq_{xx}+2G_xq_x+2fGq_x=0, \tag{17}$$

$$\begin{aligned}&G_{xx}-G_t+2gF_x-2gHq_x+2\sigma Gq_x^2+2fG_x\\&+2hGq_x+2\sigma gGq_x=0,\end{aligned} \tag{18}$$

$$\begin{aligned}&Hq_t+2fF_x-F_t-2fHq_x+2hG_x-2H_xq_x\\&+F_{xx}-Hq_{xx}+2\sigma fGq_x+2gH_x\\&+2\sigma G_xq_x+\sigma Gq_{xx}-\sigma Gq_t=0,\end{aligned} \tag{19}$$

$$\begin{aligned}&-2\sigma gHq_x+H_{xx}+2hF_x+2\sigma Hq_x^2-2hHq_x\\&+2\sigma G_xq_x+2\sigma hGq_x+2fH_x-H_t=0,\end{aligned} \tag{20}$$

$$\begin{aligned}&2hH_x-\sigma Hq_{xx}+\sigma Hq_t\\&-2\sigma H_xq_x-2\sigma fHq_x=0,\end{aligned} \tag{21}$$

$$\sigma^2Hq_x^2-\sigma hHq_x=0. \tag{22}$$

Based on (16) and (22), we have

$$g=-q_x,\quad h=\sigma q_x. \tag{23}$$

Substituting (23) into (7) and (15), we obtain

$$H=\sigma q_y,\quad G=-q_y. \tag{24}$$

Inserting (23) and (24) into (21), we have

$$f=-\frac{q_{xx}-q_t}{2q_x} \tag{25}$$

Substituting (23), (24) and (25) into (18) yields

$$F=\int-\frac{q_{xxy}q_x-q_xq_{yt}-q_{xx}q_{xy}+q_tq_{xy}}{2q_x^2}\mathrm{d}x. \tag{26}$$

Using (23) – (26) to reduce the remaining equations, we find (8) – (10), (12) – (14), (17) and (20) are satisfied identically while (11) and (19) read

$$\begin{aligned}&-2q_tq_x^2q_{xxxy}+4q_xq_{yt}q_{xx}^2+4q_{xy}q_tq_xq_{xxx}\\&-8q_{xy}q_{xx}q_xq_{xxx}-16\sigma q_{xy}q_{xx}q_x^4\\&-2q_{xt}q_{xx}q_xq_t+8q_{xy}q_{xx}q_xq_{xt}\\&-4q_{xy}q_tq_xq_{xt}+8q_{xxy}q_{xx}q_xq_t\\&+q_x^3(2q_{xxyt}-q_{xxxxy}-q_{ytt})+9q_{xy}q_{xx}^3\\&-2q_x^2q_{xy}q_{xxt}+3q_{xy}q_t^2q_{xx}\\&-q_xq_{xxy}q_t^2-16\sigma q_x^5q_{xxy}\\&+q_x^2q_{xxxx}q_{xy}+4q_{xx}q_x^2q_{xxxy}\\&-4q_{xx}q_x^2q_{xyt}+2q_tq_x^2q_{xyt}\\&-12q_tq_{xy}q_{xx}^2-9q_xq_{xxy}q_{xx}^2\\&+4q_x^2q_{xxy}q_{xxx}-4q_x^2q_{xxy}q_{xt}\\&+q_x^2q_{xy}q_{tt}+2q_x^2q_{yt}q_{xt}\\&-2q_x^2q_{yt}q_{xxx}=0,\end{aligned} \tag{27}$$

and

$$\begin{aligned}&2q_tq_{xxy}q_x-q_tq_{yt}q_x+q_{xy}q_t^2-4q_tq_{xy}q_{xx}\\&-3q_{xx}q_{xxy}q_x+2q_{xx}q_{yt}q_x+3q_{xy}q_{xx}^2\\&+\int\frac{1}{q_x^3}(-q_{xxyt}q_x^2+q_{xxy}q_{xt}q_x\\&\qquad+q_{ytt}q_x^2-q_{yt}q_{xt}q_x-q_{xyt}q_tq_x\\&\qquad-q_{xy}q_{xy}q_{tt}q_x+q_{xyt}q_{xx}q_x+q_{xy}q_{xxt}q_x\\&\qquad+2q_{xt}q_{xy}q_t-2q_{xt}q_{xy}q_{xx})\mathrm{d}xq_x^3\\&+q_{xxxy}q_x^2-q_{xyt}q_x^2+q_xq_{xy}q_{xt}\\&-q_xq_{xy}q_{xxx}+16\sigma q_{xy}q_x^4=0.\end{aligned} \tag{28}$$

Substituting (23) – (26) and the solutions of (27) and (28) into (6), we obtain exact solutions of (1).

Obviously, it is very difficult to really calculate the general solution of (27) and (28). Fortunately, in this case, one of special solutions can be expressed as

$$q=\chi(x,t)+\varphi(y), \tag{29}$$

where $\chi\equiv\chi(x,t),\varphi\equiv\varphi(y)$ are two arbitrary variable separated functions of (x,t) and y, respectively. Based on the solutions of (4), one can obtain the exact solutions of (1).

Case 1. For $\sigma < 0$, we derive the following solitary wave solutions of (1)

$$u_1 = -\frac{2\sigma\chi_x^2 \tanh^2(\sqrt{-\sigma}(\chi+\varphi)) + \sqrt{-\sigma}\tanh(\sqrt{-\sigma}(\chi+\varphi))(\chi_{xx}-\chi_t) + 2\sigma\chi_x^2}{2\chi_x\sqrt{-\sigma}\tanh(\sqrt{-\sigma}(\chi+\varphi))}, \tag{30}$$

$$v_1 = -\frac{\sigma\varphi_y[\tanh^2(\sqrt{-\sigma}(\chi+\varphi))+1]}{\sqrt{-\sigma}\tanh(\sqrt{-\sigma}(\chi+\varphi))}, \tag{31}$$

$$u_2 = -\frac{2\sigma\chi_x^2 \coth^2(\sqrt{-\sigma}(\chi+\varphi)) + \sqrt{-\sigma}\coth(\sqrt{-\sigma}(\chi+\varphi))(\chi_{xx}-\chi_t) + 2\sigma\chi_x^2}{2\chi_x\sqrt{-\sigma}\coth(\sqrt{-\sigma}(\chi+\varphi))}, \tag{32}$$

$$v_2 = -\frac{\sigma\varphi_y[\coth^2(\sqrt{-\sigma}(\chi+\varphi))+1]}{\sqrt{-\sigma}\coth(\sqrt{-\sigma}(\chi+\varphi))}, \tag{33}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

Case 2. For $\sigma > 0$, we obtain the following periodic wave solutions of (1)

$$u_3 = -\frac{2\sqrt{\sigma}\chi_x^2 \tan^2(\sqrt{\sigma}(\chi+\varphi)) + \tan(\sqrt{\sigma}(\chi+\varphi))(\chi_{xx}-\chi_t) - 2\sqrt{\sigma}\chi_x^2}{2\chi_x\tan(\sqrt{\sigma}(\chi+\varphi))}, \tag{34}$$

$$v_3 = -\frac{\sqrt{\sigma}\varphi_y[\tan^2(\sqrt{\sigma}(\chi+\varphi))-1]}{\tan(\sqrt{\sigma}(\chi+\varphi))}, \tag{35}$$

$$u_4 = \frac{2\sqrt{\sigma}\chi_x^2 \cot^2(\sqrt{\sigma}(\chi+\varphi)) + \cot(\sqrt{\sigma}(\chi+\varphi))(\chi_t-\chi_{xx}) - 2\sqrt{\sigma}\chi_x^2}{2\chi_x\tan(\sqrt{\sigma}(\chi+\varphi))}, \tag{36}$$

$$v_4 = \frac{\sqrt{\sigma}\varphi_y[\cot^2(\sqrt{\sigma}(\chi+\varphi))-1]}{\cot(\sqrt{\sigma}(\chi+\varphi))}, \tag{37}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

Case 3. For $\sigma = 0$, we derive the following variable-separable solution of (1)

$$u_5 = -\frac{\chi_{xx}(\chi+\varphi) - \chi_t(\chi+\varphi) - 2\chi_x^2}{2\chi_x(\chi+\varphi)}, \tag{38}$$

$$v_5 = \frac{\varphi_y}{\chi+\varphi}, \tag{39}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

All this is along meanwhile well-known lines to find solutions of NPDEs.

3. Stochastic Fractal Solitons and Regular Fractal Solitons of the (2+1)-Dimensional BLP System

In this section, we mainly discuss the solitary wave solutions, namely Case 1. Owing to the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in this case, the physical quantities u and v may possess rich localized structures when selecting the functions $\chi(x,t)$ and $\varphi(y)$ appropriately. For simplicity in the following discussion, we merely analyze their potentials u_{1y} or v_{1x} determined by (30) or (31) and rewrite them in a simple form, namely

$$U \equiv u_{1y} = v_{1x} = \frac{\sigma\chi_x\varphi_y[\tanh^2(\sqrt{-\sigma}(\chi+\varphi))-1]^2}{\tanh^2(\sqrt{-\sigma}(\chi+\varphi))}. \tag{40}$$

3.1. Stochastic Fractal Dromions and Lumps

Now, we discuss some localized coherent solitons with fractal properties. It is well-known that there are some lower-dimensional stochastic fractal functions, which may be used to construct higher-dimensional stochastic fractal dromion and lump excitations. For instance, one of the most well-known stochastic fractal functions is the Weierstrass function

$$\Re \equiv \sum_{k=0}^{N}[\lambda^{(s-2)k}\sin(\lambda^k\xi)], \quad N \longrightarrow \infty, \tag{41}$$

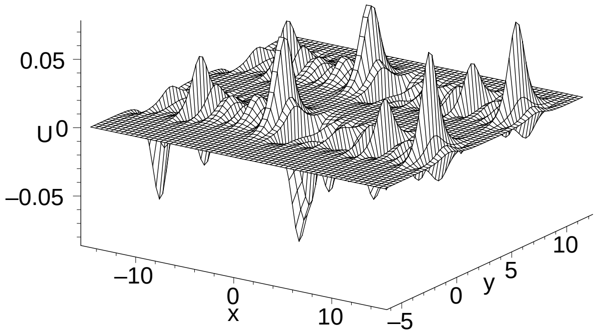

Fig. 1. A plot of a typical stochastic fractal dromion solution determined by (40) with the choices (41) and (42).

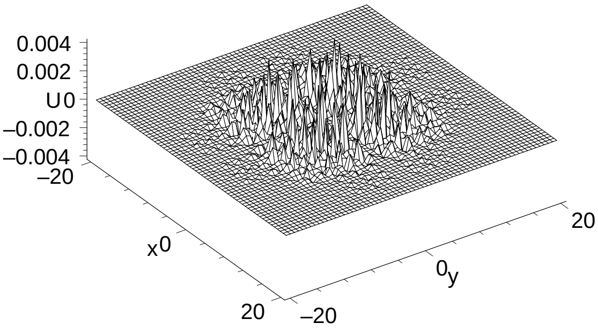

Fig. 2. A plot of a typical stochastic fractal lump solution determined by (40) with the choices (41), (43) and $N = 30$.

where λ, s are constants and the independent variable ξ may be a suitable function of $x+ct$ and/or y, say $\xi = x+ct$ and $\xi = y$ in the functions of χ and φ for the following choise

$$\begin{aligned}\chi &= 3 - 0.1\Re(x+ct)\tanh[3(x+t)],\\ \varphi &= 0.06\tanh(y) + 0.1\tanh(y-8),\end{aligned} \tag{42}$$

or

$$\begin{aligned}\chi &= -1.5 + 0.1\exp[-0.02(\Re(x+ct)+x)(x+t)],\\ \varphi &= -1.5 + 0.1\exp[-0.02(\Re(y)+y)y].\end{aligned} \tag{43}$$

If the Weierstrass function is included in soliton solutions, then we can derive some special stochastic fractal dromions and lumps.

Figures 1 and 2 respectively show plots of typical stochastic fractal dromion and lump solutions determined by (40) with the choices (41), (42) and (43) $(\lambda = s = 1.5, c = -\sigma = 1)$ at $t = 0$. From Fig. 1, one can find that the amplitudes of the multi-dromion are irregularly changed. Similarly, the shapes of the multi-lump in Fig. 2 are also altered irregularly.

3.2. Regular Fractal Dromions and Lumps with Self-similar Structures

In addition to stochastic fractal dromions and fractal lumps, there may exist some regular fractal localized excitations. We know that it is very difficult to find some appropriate functions which can be used to depict regular fractal patterns possessing self-similar structures. Fortunately, in a recent study, we have found many lower-dimensional piecewise smooth functions with fractal properties, which can be used to construct exact solutions of higher-dimensional soliton systems, which also possess fractal structures, such as some piecewise smooth functions of sine function, cosine function, Jacobian function and Bessel function. For example, when choosing χ and φ in solution (40) to be

$$\begin{aligned}\chi &= 1 + \frac{|x+t|\{\text{Bessel-}J[0,\ln(x+t)^2]\}^2}{1+(x+t)^4},\\ \varphi &= 1 + \frac{|y|\{\text{Bessel-}J[0,\ln(y^2)]\}^2}{(1+y^4)},\end{aligned} \tag{44}$$

we can derive a fractal lump solution with self-similar structure.

Figure 3(a) shows a plot of the special type of fractal lump structure for the potential U given by (40) with the choice (44) and $\sigma = -1$ at $t = 0$. From Fig. 3, we notice that the lump structure possesses a self-similar fractal property. Near the center in Fig. 3(a) there are many peaks which are distributed in a fractal manner. In order to observe the self-similar structure of the fractal lump more clearly, one may enlarge a small region near the center of Figure 3(a). Figures 3(b) and 3(c) show plots of the self-similar structure of the fractal lump in the region $\{x \in [-0.00058,\ 0.00058],\ y \in [-0.00058,\ 0.00058]\}$ and $\{x \in [-0.0000052, 0.0000052], y \in [-0.0000052, 0.0000052]\}$. From Fig. 3(b) and 3(c), one can easily find the self-similar structure of the fractal lump. Figure 3(d) shows the density of the fractal structure of the lump related to Fig. 3(b) in the region $\{x \in [-0.00058, 0.00058],\ y \in [-0.00058,\ 0.00058]\}$. If we enlarge a smaller region near the center of Fig. 3(d), we find a totally similar structure to that presented in Figure 3(d).

4. Summary and Discussion

Usually, the mapping approach is only presented for finding travelling wave solutions of nonlinear elliptic equations (NEEs) [27–29], such as the function

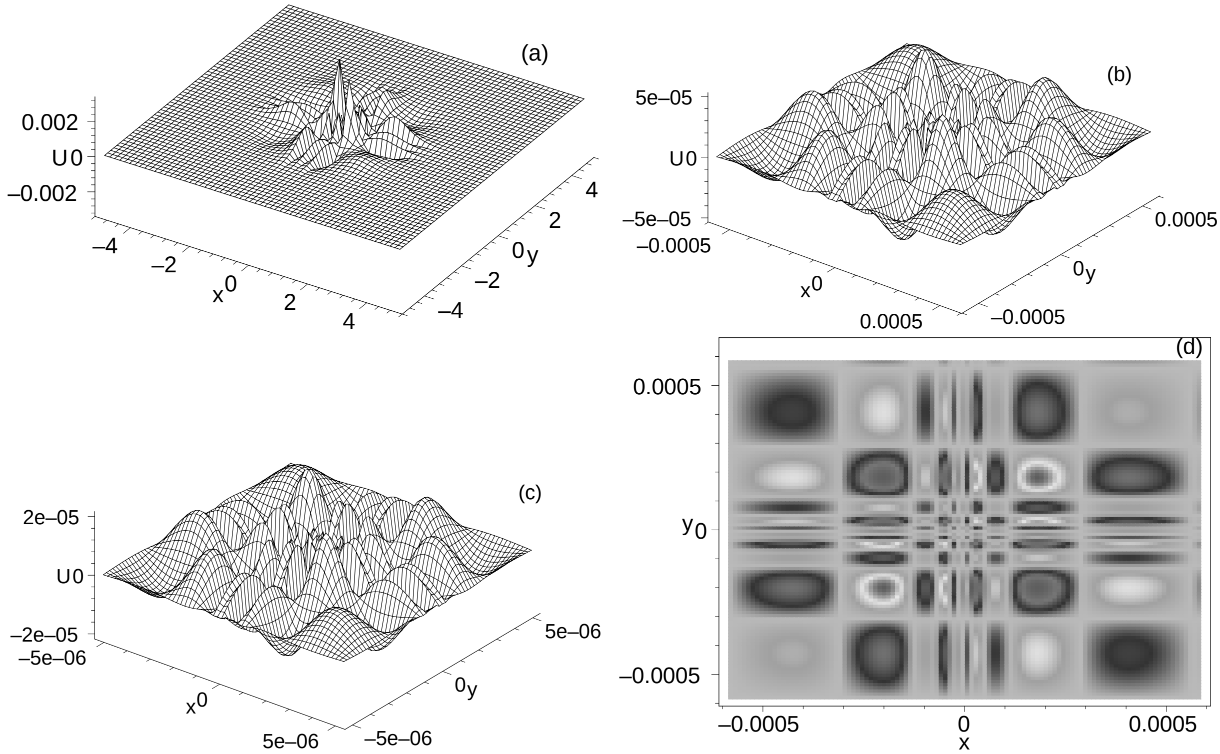

Fig. 3. (a) A fractal lump structure for the potential U given by (40) with the choice (44) and $\sigma = -1$ at time $t = 0$. (b) Self-similar structure of the fractal lump related to (a) in the region $\{x =\in [-0.00058,\ 0.00058],\ y \in [-0.00058,\ 0.00058]\}$. (c) Self-similar structure of the fractal lump related to (a) in the region $\{x \in [-0.0000052,\ 0.0000052],\ y \in [-0.0000052,\ 0.0000052]\}$. (d) Density of the fractal structure of the lump related to (b) in the region $\{x \in [-0.00058,\ 0.00058],\ y \in [-0.00058,\ 0.00058]\}$.

q in (29) selected to be $\chi = ax + ct,\ \varphi = by$. However in this paper, with the help of the extended mapping approach, a new type of variable separation solution with two arbitrary functions of the BLP system is derived. Based on the derived solitary wave excitation and by choosing several appropriate functions, we have found some new localized excitations that possess regular fractal and irregular, chaotic fractal properties. Because of some important applications of the Bessel function and Weierstrass function in natural science, we are sure that these new fractal solitons would be significant, since fractals not only belong to the realms of mathematics or computer graphics, but also exist nearly everywhere in nature, such as in fluid turbulence, crystal growth patterns, human veins, fern shapes, galaxy clustering, cloud structures and in numerous other examples. Traditionally, it is believed that fractals are opposite objects to solitons in nonlinear science since solitons are representatives of integrable systems while fractals typically represent the behaviour of nonintegrable systems. However, this assessment may be somewhat absolute. The main reason is that, when talking about a system to be integrable, one has to emphasize an important fact: one has to point out under which special meaning the system is integrable. For example, one say a model is Painlevé integrable if it possesses Painlevé property, and a system is Lax or IST (Inverse Scattering Transformation) integrable if it has a Lax pair and can be solved by the IST approach. Nevertheless, a system integrable in some special sense like the Lax integrability may not be integrable in another sense such as Painlevé integrability. Consider, for instance, the (2+1)-dimensional dispersive long wave system,

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \tag{45}$$

$$v_t + u_x + v u_x + u v_x + u_{xxy} = 0, \tag{46}$$

which has first been obtained by Boiti *et al.* as a compatibility condition for a “weak” Lax pair. Though the system is Lax or ISI integrable, it does not pass the Painlevé test, which means it is not Painlevé integrable [15].

Along this line and considering the fact that some lower-dimensional characteristic functions may exist in the exact solutions of many (2+1)-dimensional systems, one can elucidate deduce that fractals in higher-dimensional integrable physical models can be a quite universal phenomenon. Why do the localized excitations possess such kinds of fractal properties? If one considers the boundary and/or initial conditions of the fractal solutions obtained here, one can straightforwardly find that the initial and/or boundary conditions possess a fractal property. These fractal properties of the localized excitations for an integrable model essentially come from certain "nonintegrable" fractal boundary and/or initial conditions.

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